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CERTAIN CONTACT PROBLEMS FOR A HALF-SPACE REINFORCED BY ELASTIC GUSSETS

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Contact problems for bodies with elastic reinforcements in the form of gussets (stringers) of slight thickness are directly connected with questions of load transmission from the gussets to elastic bodies, which are important for engineering practice. Various plane problems have been investigated in many papers. For example, two fundamental problems on the transmission of a load from a gusset infinite in both directions to a semi-infinite and infinite plate have been examined in [1]. The model of a one-dimensional elastic continuum of the gusset is taken as the fundamental physical model. A number of papers devoted to various extensions and modifications of the fundamental Melan problems was later executed within the scope of the physical assumptions in [1]. A sufficiently complete bibliography of these papers is contained in [2, 3].

The papers [4 – 7] are devoted to giving a foundation for the model of the one-dimensional model of the elastic continuum of the gusset and to investigating some other contact problems for a half-plane with elastic gussets. While the domain of plane contact problems for bodies with elastic gussets of slight thickness has been developed sufficiently well, the domain of three-dimensional contact problems for bodies with elastic gussets of slight cross section has hardly ever been investigated , and the authors know of no papers in this area where a rigorous solution of such problems would be presented. In some sense, paper [8], referring to questions of determining the contact stresses on the lateral surface of a cylindrical rod imbedded in an elastic space or half-space, is an exception. Such a situation in the area of three-dimensional problems is explained by the fact that significant mathematical difficulties are encountered in their solution. Moreover, the model of one-dimensional elastic continuum for the gusset in combination with the model of contact along a line is not directly applicable in the formulation of three-dimensional contact problems for bodies with elastic gussets of small cross section. The model of contact along an area, when it is assumed that the stresses in the contact zone are distributed uniformly in the transverse direction, also does not correspond completely to reality.

In contrast to the case of plane problems, a new approach is proposed herein to the formulation of three-dimensional contact problems for a half-space reinforced by elastic gussets of small cross section. Three kinds of contact problems are then examined, namely, problems when the elastic half-space is reinforced on some part of its boundary by an infinitely long gusset, a semi-infinite gusset, and a gusset of finite length. In the proposed formulation, the solution of these problems reduces to solving integro-differential equations with kernels expressible by complete elliptic integrals of the first and second kinds, under definite boundary conditions. An effective method of solving these equations is proposed.

1. Formulation of the problems and derivation of the governing equations. Let an elastic half-space be reinforced on some part of its boundary plane by an elastic gusset of rectangular cross section of infinite length (Fig. 1). Let us consider the area F of the gusset cross section to be sufficiently small, i.e. the thickness h and the half-width δ of the gusset to be sufficiently small. It is required to determine the

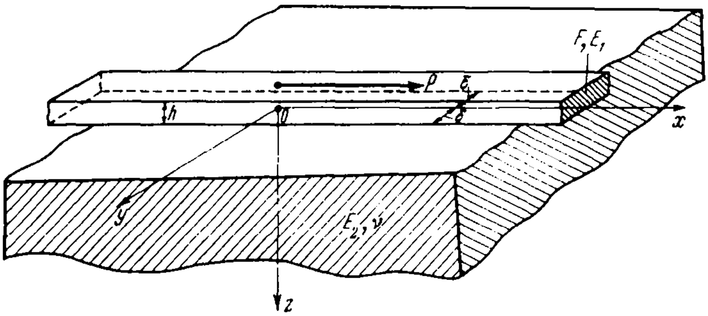

Fig. 1

distribution law of the contact stresses in the strip connecting the gusset to the half-space when a concentrated force P directed along the gusset axis, acts at some point of the upper face of the gusset. As in [1, 4], let us assume that the gusset is bent negligibly little because of the smallness of the gusset thickness h, i.e. the normal pressure of the gusset on the half-space can be neglected in the area of contact. A sufficiently complete foundation for this assumption is presented in [7] for the plane problem case. Furthermore, let us assume that the half-width δ is so small that the tangential stresses τ_{yz} can be neglected in the contact zone.

It follows from these assumptions that only tangential contact stresses $\tau_{xz}(x, y)$ act in the strip connecting the gusset to the half-space. If the model of contact along the line is taken, and it is assumed that these stresses are concentrated along the middle line of the contact strip, i.e. along the abscissa axis, and are applied to the half-space, then we can arrive at the following deduction. As is known [9], the displacements of a boundary point of the half-space with coordinates (x, y) along the Ox axis from a force Q concentrated at the origin and directed along the same axis are given by

$$u(x, y) = \frac{(1-\nu^2)Q}{\pi E_2} \frac{1}{\sqrt{x^2+y^2}} \left(1 + \frac{\nu}{1-\nu} \frac{x^2}{x^2+y^2}\right) \tag{1.1}$$

where E_2 is the elastic modulus, ν is the Poisson ratio of the material of the half-space. In particular, for points on the abscissa axis we have

$$u(x, 0) = \frac{(1+\nu)Q}{\pi E_2} \frac{1}{|x|}$$

In the case of a load of intensity $q(x)$ distributed along some segment $[a, b]$ of the Ox axis, we obtain

$$u(x, 0) = \frac{1+\nu}{\pi E_2} \int_a^b \frac{q(s)\, ds}{|x - s|}$$

However, this integral does not converge in either the ordinary sense of the convergence of improper integrals at the point $s = x$, or in the sense of the Cauchy principal value. The difficulty with convergence of the integral, caused by assuming the model of contact along a line, shows that this model is inapplicable in the case of the problem under discussion.

Now, let us assume that the stresses $\tau_{xz}(x, y)$ are distributed uniformly in the transverse direction in the whole contact strip $-\infty < x < \infty$, $-\delta < y < \delta$, i.e. we assume that the model of contact on an area holds. Then $\tau_{xz}(x, y) = \tau_{xz}(x)$, and on the basis of (1.1) we have

$$u(x, 0) = \frac{2(1-\nu^2)}{\pi E_2} \int_{-\infty}^{\infty} h(x - s)\, \tau_{xz}(s)\, ds \qquad (1.2)$$

$$h(x) = \ln \frac{1}{|x|} + \ln(\delta + \sqrt{\delta^2 + x^2}) + \frac{\nu}{1-\nu}\left[1 - \frac{x^2}{\sqrt{\delta^2 + x^2}\,(\delta + \sqrt{\delta^2 + x^2})}\right] \qquad (-\infty < x < \infty) \qquad (1.3)$$

These latter formulas show that if the assumption of uniformity of the contact tangential stress distribution in the transverse direction is introduced, then the displacements of points of the middle line of the contact strip are completely definite quantities along the same line. This assumption holds when the gusset is partially fastened to the half-space in the transverse direction. Namely, when the base of the gusset is fastened to the half-space just along the strip $|y| < \delta_1$ $(\delta_1 < \delta)$ so that the parts $\delta_1 < |y| < \delta$ of the gusset base are free of contact stresses; the quantities $\tau_{xz}(x, y)$ are finite. Because of the continuity in the change in these quantities and the smallness of the width of the contact domain, it can be considered that the contact stresses $\tau_{xz}(x, y)$ in the transverse direction are uniformly distributed.

This assumption will not hold when the gusset base is fastened completely to the half-space since the contact stresses $\tau_{xz}(x, y)$ become infinite for $y = \pm\delta$. In other words, the lines $y = \pm\delta$ are singular lines for the stresses $\tau_{xz}(x, y)$. In order to clarify the exact form of the singularities of these stresses on the lines $y = \pm\delta$, we proceed as follows. The displacements of the boundary points of an elastic half-space along the Ox axis from the stresses $\tau_{xz}(x, y)$ distributed over the contact domain $-\infty < x < \infty$, $-\delta < y < \delta$ are given according to (1.1) by

$$u(x, y) = \frac{1-\nu^2}{\pi E_2} \int_{-\infty}^{\infty} d\xi \int_{-\delta}^{\delta} \left\{ \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} + \frac{\nu}{1-\nu} \frac{(x-\xi)^2}{[(x-\xi)^2 + (y-\eta)^2]^{3/2}} \right\} \tau_{xz}(\xi, \eta)\, d\eta$$

Applying the Fourier transform in the variable x to both sides of this equality, we obtain

$$\bar{u}(\lambda, y) = \frac{2(1+\nu)}{\pi E_2} \int_{-\delta}^{\delta} K_0(\lambda|y-\eta|) - \nu\lambda|y-\lambda| K_1(\lambda|y-\eta|)\, \bar{\tau}_{xz}(\lambda, \eta)\, d\eta \qquad (1.4)$$

$$\bar{u}(\lambda, y) = \int_{-\infty}^{\infty} u(x, y)\, e^{i\lambda x}\, dx, \qquad \bar{\tau}_{xz}(\lambda, y) = \int_{-\infty}^{\infty} \tau_{xz}(x, y)\, e^{i\lambda x}\, dx$$

and $K_0(x)$ and $K_1(x)$ are the known Macdonald functions. Taking into account the series expansions for these latter functions, we can write

$$K_0(\lambda \mid y-\eta \mid) - \nu\lambda \mid y-\eta \mid K_1(\lambda \mid y-\eta \mid) =$$
$$= \ln \mid y-\eta \mid^{-1} + R(\mid y-\eta \mid)$$

where $R(x)$ $(-\infty < x < \infty)$ is a definite continuous function.

Such a representation of the kernel in (1.4) permits the assertion that singularities of the function $\tau_{xz}(x, y)$ are of square root type on the lines $y = \pm\delta$, as in the case of the classical contact problem of the impression of a stamp in an elastic half-plane without taking account of friction. This fact can be proved directly by relying on the results about the behavior of a Cauchy-type integral at the endpoints of the lines of integration from [10], as has been done in [6, 7].

The reasoning cited suggests that the contact stress distribution in the transverse direction in the strip connecting the gusset to the elastic half-space can be considered exactly as has been obtained on the basis of solving the mentioned plane contact problem. This assumption is also taken in the problem of impressing a narrow beam in an elastic half-space [11]. It is analogous to the assumption on which the theory of a narrow finite-span wing is constructed.

Thus, in place of the hypothesis of uniformity of the contact stress distribution $\tau_{xz}(x, y)$ in the transverse direction, we assume that the distribution of these stresses is given by

$$\tau_{xz}(x, y) = \tau(x) / (\pi \sqrt{\delta^2 - y^2}) \tag{1.5}$$

where $\tau(x)$ is the stress per unit length of the gusset to be determined. Furthermore, we will assume that the axial stresses $\sigma_x^{(1)}$ are uniformly distributed over the cross-sectional area in any transverse section of the gusset. Moreover, we shall consider the contact stresses under the gusset to be concentrated along the middle line of the contact strip. Therefore, the hypotheses reduce to the fact that the model of a one-dimensional elastic continuum for the gusset in combination with the model of contact along the area for a half-space holds when the contact stress distribution law is given by (1.5).

Let us now proceed to deduce the fundamental governing equations. To this end, let us consider the equilibrium of an infinitesimal element of the gusset between the planes x and $x + dx$. On the basis of the assumptions made we can write

$$F \frac{d\sigma_x^{(1)}}{dx} - \int_{-\delta}^{\delta} \tau_{xz}(x, y)\, dy + P\delta(x) = 0$$

where $\tau_{xz}(x, y)$ are the known contact stresses acting in the strip connecting the gusset to the half-space and applied to the gusset, and $\delta(x)$ is the known Dirac function. Taking (1.5) and Hooke's law into account, we have

$$\frac{d^2 u^{(1)}}{dx^2} = \frac{\tau(x) - P\delta(x)}{FE_1} \tag{1.6}$$

where E_1 is the elastic modulus of the gusset material, and $u^{(1)}(x)$ is the displacement of points of the middle line of the gusset base along the same axis.

On the other hand, the displacements $u^{(2)}(x, 0)$ of the boundary points of the elastic half-space on the middle line of the contact domain, which occur along the same line due to the contact stresses $\tau_{xz}(x, y)$ applied to the half-space are according to (1.1)

$$u^{(2)}(x, 0) = \frac{1-\nu^2}{\pi E_2} \int_{-\infty}^{\infty} d\xi \int_{-\delta}^{\delta} \frac{1}{\sqrt{(x-\xi)^2+\eta^2}} \left[1 + \frac{\nu}{1-\nu} \frac{(x-\xi)^2}{(x-\xi)^2+\eta^2}\right] \tau_{xz}(\xi, \eta)\, d\eta$$

Taking account of (1.5), after some manipulation we obtain

$$u^{(2)}(x,0) = \frac{2(1-\nu^2)}{\pi^2 E_2} \int_{-\infty}^{\infty} \left[k_1(x-s) + \frac{\nu}{1-\nu} l_1(x-s)\right] \tau(s)\, ds \qquad (1.7)$$

$$k_1(x) = \int_0^{\delta} \frac{d\eta}{\sqrt{(x^2+\eta^2)(\delta^2-\eta^2)}} = \frac{1}{|x|} K\left(\frac{i\delta}{|x|}\right)$$

$$l_1(x) = x^2 \int_0^{\delta} \frac{d\eta}{(x^2+\eta^2)^{3/2}\sqrt{\delta^2-\eta^2}} \qquad (-\infty < x < \infty)$$

where $K(k)$ is the complete elliptic integral of the first kind of modulus k. Evidently

$$l_1(x) = -x \frac{dk_1(x)}{dx} \qquad (1.8)$$

then we find after elementary computations

$$k_1(x) = \frac{1}{\sqrt{\delta^2+x^2}} K\left(\frac{\delta}{\sqrt{\delta^2+x^2}}\right)$$

Using (1.8), we obtain by the formula for the differentiation of the function $K(k)$ [12]

$$l_1(x) = \frac{1}{\sqrt{\delta^2+x^2}} E\left(\frac{\delta}{\sqrt{\delta^2+x^2}}\right), \qquad E(k) = \int_0^1 \sqrt{\frac{1-k^2x^2}{1-x^2}}\, dx \qquad (1.9)$$

where $E(k)$ is the complete elliptic integral of the second kind of modulus k. The condition

$$u^{(1)}(x) = u^{(2)}(x, 0)$$

must be satisfied on the middle line of the contact domain, and in combination with (1.6) it reduces the problem of determining the unknown contact stress $\tau(x)$ to the solution of the following integro-differential equation:

$$\frac{d^2}{dx^2} \int_{-\infty}^{\infty} \left[k(x-s) + \frac{\nu}{1-\nu} l(x-s)\right] \tau(s)\, ds = \mu^2 \left[\tau(x) - P\delta(x)\right] \qquad (1.10)$$

$$\mu^2 = \pi^2 E_2 \left[4(1-\nu^2) h E_1\right]^{-1}$$

$$k(x) = \delta k_1(x), \qquad l(x) = \delta l_1(x)$$

where in conformity with (1.8) and (1.9)

$$k(x) = \frac{\delta}{\sqrt{\delta^2+x^2}} K\left(\frac{\delta}{\sqrt{\delta^2+x^2}}\right), \quad l(x) = \frac{\delta}{\sqrt{\delta^2+x^2}} E\left(\frac{\delta}{\sqrt{\delta^2+x^2}}\right) \qquad (1.11)$$

Furthermore, we find from the equilibrium condition of the gusset that the solution of the integro-differential equations (1.10) must satisfy the relation

$$\int_{-\infty}^{\infty} \tau(x)\, dx = P$$

Therefore, the solution of the contact problem for a half-space reinforced by an elastic infinite gusset of small cross section reduces, under the physical and geometric hypotheses made above, to the solution of the integro-differential equation (1.10) with a kernel expressed by (1.11). This integro-differential equation is evidently equivalent to the integro-differential equation

$$\frac{d}{dx} \int_{-\infty}^{\infty} \left[k(x-s) + \frac{\nu}{1-\nu}\, l(x-s)\right] \varphi'(s)\, ds = \mu^2 \varphi(x) - \mu^2 P H(x) \qquad (1.12)$$

where $H(x)$ is the known Heaviside function

$$\varphi(x) = \int_{-\infty}^{x} \tau(s)\, ds$$

Writing the integro-differential equation (1.10) thus is outwardly similar to analogous equations encountered in plane contact problems for the bodies with elastic gussets, particularly the equations for the fundamental Melan problems.

Turning to the case of a semi-infinite gusset (Fig. 2), we find completely analogously

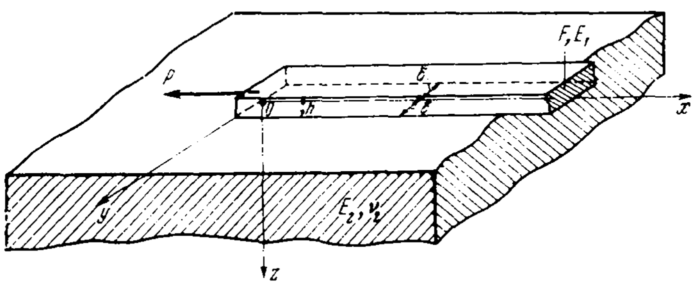

Fig. 2

that the solution of the appropriate contact problem reduces to the solution of the integro-differential equation

$$\frac{d^2}{dx^2} \int_{0}^{\infty} \left[k(x-s) + \frac{\nu}{1-\nu}\, l(x-s)\right] \tau(s)\, ds = \mu^2 \tau(x) \qquad (1.13)$$

under the condition

$$\int_{0}^{\infty} \tau(x)\, dx = P \qquad (1.14)$$

Equation (1.13) together with condition (1.14) is equivalent to the integro-differential equation

$$\frac{d}{dx} \int_{0}^{\infty} \left[k(x-s) + \frac{\nu}{1-\nu}\, l(x-s)\right] \varphi'(s)\, ds = \mu^2 \varphi(x) \qquad (1.15)$$

under the boundary conditions

$$\varphi(0) = P, \qquad \varphi(\infty) = 0 \qquad (1.16)$$

$$\varphi(x) = P - \int_{0}^{x} \tau(s)\, ds$$

where $\tau(x)$ is an unknown contact stress acting in the strip connecting the semi-infinite gusset to the elastic half-space.

Finally, the solution of the contact probem for a half-space reinforced on its boundary by a finite gusset of small cross section (Fig. 3) reduces to the solution of the integro-differential equation

$$\frac{d}{dx}\int_{-a}^{a}\left[k(x-s)+\frac{\nu}{1-\nu}l(x-s)\right]\varphi'(s)\,ds=\mu^2\varphi(x) \qquad (1.17)$$

under the boundary conditions

$$\varphi(-a)=0, \qquad \varphi(a)=P \qquad (1.18)$$

$$\varphi(x)=\int_{-a}^{x}\tau(s)\,ds$$

where $\tau(x)$ is an unknown contact stress under the elastic gusset.

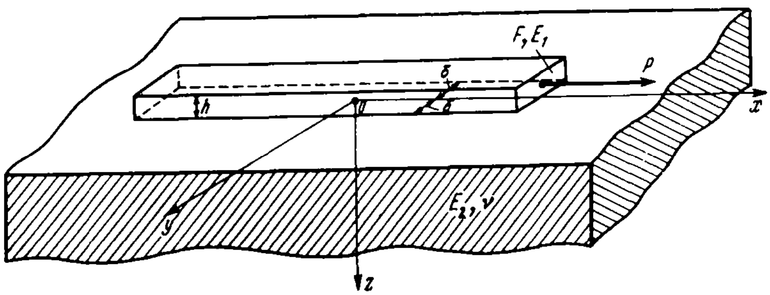

Fig. 3

The solution of the posed problems under the assumption of uniformity of the contact stress distribution in the transverse direction of the gusset is evidently reduced to the very same equations on the basis of (1.2), but only with kernels in the form of the function $h(x)$ form (1.3). However, we shall not consider these equations herein.

Let us now investigate the structure of the kernels of the mentioned integro-differential equations. It is easy to see that the complete elliptic integral of the first kind $K(k)$ becomes infinite for $k=1$, and therefore, the function $k(x)$ from (1.11) has a singularity at $x=0$. Let us elucidate the form of this singularity. According to (1.8) and (1.11), for $x=0$ the function $k(x)$ has a logarithmic singularity. Let us represent this function as

$$k(x)=\ln\frac{1}{|x|}+r(x) \qquad (1.19)$$

$$r(x)=\frac{\delta}{\sqrt{\delta^2+x^2}}K\left(\frac{\delta}{\sqrt{\delta^2+x^2}}\right)-\ln\frac{1}{|x|} \qquad (1.20)$$

where the function $r(x)$, on the basis of the above, is continuous and has continuous derivatives, at least to second order inclusive, on the whole real axis. As regards the function $l(x)$ expressed by the second of formulas (1.11), it is continuous on the whole real axis and its values fill the segment [0, 1]. Furthermore we have

$$k(x)+\frac{\nu}{1-\nu}l(x)=k(x)\left[1+\frac{\nu}{1-\nu}\frac{l(x)}{k(x)}\right]$$

Evidently $l(0)/k(0) = 0$, and for any $x \neq 0$ a δ can be selected so small and independent of x, that the function $l(x)/k(x)$ would be arbitrarily close to unity for all $x \neq 0$. Then, to arbitrarily high accuracy we can assume

$$k(x) + \frac{\nu}{1-\nu} l(x) = k(x)\chi(x)$$

$$\chi(x) = \begin{cases} 1 & (x = 0) \\ 1/1-\nu & (x \neq 0) \end{cases}$$

provided that δ is sufficiently small. In other words, for sufficiently narrow gusset this equality will differ as little as desired from the true. Let us note that this result can be obtained at once if we set

$$\frac{x^2}{x^2+y^2} = \begin{cases} 1 & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

for sufficiently small δ in the influence function (1.1).

The elucidated investigation of the kernel permits replacement of the integro-differential equations (1.10) and (1.13) under the condition (1.14), and (1.17) under the boundary conditions (1.18) obtained to determine the contact stresses under infinite, semi-infinite, and finite gussets of sufficiently small width, respectively, by integro-differential equations

$$\frac{d^2}{dx^2} \int_{-\infty}^{\infty} k(x-s)\,\tau(s)\,ds = c^2[\tau(x) - P\delta(x)] \qquad (1.10')$$

$$\frac{d^2}{dx^2} \int_{0}^{\infty} k(x-s)\,\tau(s)\,ds = c^2\tau(x) \qquad (1.13')$$

under the condition

$$\int_{0}^{\infty} \tau(x)\,dx = P \qquad (1.14')$$

and the integro-differential equation

$$\frac{d}{dx} \int_{-a}^{a} k(x-s)\,\varphi'(s)\,ds = c^2\varphi(x) \qquad (1.17')$$

under the boundary conditions

$$\varphi(-a) = 0, \qquad \varphi(a) = P \qquad (1.18')$$

Here

$$c^2 = \mu^2(1-\nu) = \pi^2 E_2 [4(1+\nu) h E_1]^{-1}$$

Analogous equations corresponding to the integro-differential equations (1.12) and (1.15) under the boundary conditions (1.16) in the sense mentioned can also be obtained. If the derivative symbol is introduced under the integral sign in these latter equations and in (1.17'), and the form of the function $k(x)$ is taken into account, then integro-differential equations of analogous structure to the corresponding equations for plane contact problems are obtained. The difference is just that the kernel of the equations in the cases we considered consists of a Cauchy kernel and some continuous function taking account of the three-dimensional effect of the problems under consideration.

Let us note that the kernal $k(x-s)$ generated by the functions $k(x)$ is a singular kernel for $-a < x,\ s < a$. An investigation of this kernel is contained in [11].

Let us also note that the boundary conditions (1.18), as well as the condition (1.14), express the fact that the loads applied to the gussets are transmitted completely to the base without concentrated components. This can be proved by a method completely analogous to that mentioned in [3].

2. Case of an infinite gusset. The solution of the integro-differential equation (1.10') is easily obtained by using the Fourier transform. Performing the transformation, we arrive at the algebraic equation

$$[c^2 - \lambda^2 K(\lambda)]\, T(\lambda) = -c^2 P$$

$$K(\lambda) = \int_{-\infty}^{\infty} k(x)\, e^{i\lambda x}\, dx, \qquad T(\lambda) = \int_{-\infty}^{\infty} \tau(x)\, e^{i\lambda x}\, dx \tag{2.1}$$

Hence

$$\tau(x) = -\frac{c^2 P}{\pi} \int_0^{\infty} \frac{\cos \lambda x\, d\lambda}{c^2 + \lambda^2 K(\lambda)} \tag{2.2}$$

Using the known relationships ([12], p. 443, formula 3.773.6, p. 716, formula 6.592.7), we find

$$K(\lambda) = \pi I_0(^1/_2\lambda\delta)\, K_0(^1/_2\lambda\delta) \quad (\lambda > 0)$$
$$K(-\lambda) = K(\lambda) \tag{2.3}$$

where $I_0(x)$ is the modified Bessel function of the first kind. The following asymptotic formulas [12]:

$$I_0(x) \approx \sqrt{\frac{1}{2\pi x}}\, e^x, \qquad K_0(x) \approx \sqrt{\frac{\pi}{2x}}\, e^{-x} \tag{2.4}$$

are valid for the functions $I_0(x)$ and $K_0(x)$ for large positive values of the argument x.

Let us introduce the function

$$f(\lambda) = c^2 + \lambda^2 K(\lambda)$$

This function is evidently continuous on the whole real axis and is strictly positive, and $f(0) = c^2$. Moreover, in conformity with (2.3)

$$f(\lambda) \approx c^2 + \pi\lambda / \delta \qquad \lambda \to +\infty \tag{2.5}$$

Let us introduce the function $H(\lambda)$ related to $f(\lambda)$ by means of

$$H(\lambda) = \frac{c^2 + \pi\lambda/\delta}{f(\lambda)} - 1 = \frac{c^2 + \pi\lambda/\delta}{c^2 + \lambda^2 K(\lambda)} - 1$$

where

$$H(0) = H(\infty) = 0 \tag{2.6}$$

Now turning to (2.2), let us represent it as

$$\tau(x) = -\frac{c^2 P}{\pi} \int_0^{\infty} \frac{H(\lambda) \cos \lambda x}{c^2 + \pi\lambda/\delta}\, d\lambda - \frac{c^2 P}{\pi} \int_0^{\infty} \frac{\cos \lambda x\, d\lambda}{c^2 + \pi\lambda/\delta}$$

The second integral in this latter equality is expressed by the formula

$$\int_0^{\infty} \frac{\cos \lambda x\, d\lambda}{c^2 + \pi\lambda/\delta} = -(\cos qx \operatorname{Ci} qx + \sin qx \operatorname{Si} qx)$$

$$(q = \delta c^2 / \pi)$$

where $\operatorname{Si}(x)$, $\operatorname{Ci}(x)$ are the sine and cosine intergal functions of x. Then we will have

$$\tau(x) = -\frac{c^2 P}{\pi} \int_0^{\infty} \frac{H(\lambda) \cos \lambda x}{c^2 + \pi\lambda/\delta}\, d\lambda + \frac{c^2 \delta P}{\pi} (\cos qx \operatorname{Ci} qx + \sin qx \operatorname{Si} qx) \tag{2.7}$$

It follows from (2.5) and (2.6) that as $x \to \infty$ the integral in (2.7) tends to zero fairly rapidly, and therefore, the behavior of the contact stresses under the gusset is governed completely by the second member of this formula for large x. This term, i.e. the function

$$p(x) = \frac{c^2\delta P}{\pi^2}(\cos qx \operatorname{Ci} qx + \sin qx \operatorname{Si} qx) \tag{2.8}$$

is the known Melan solution [1] to the accuracy of the factor q. Therefore, the contact stresses under an infinite gusset decrease according to the law of the Melan function (2.8) for large x. On the other hand, evidently $\tau(0) = \infty$.

3. Case of a semi-infinite gusset. A solution of the integro-differential equation (1.13') under the condition (1.14') can be constructed by the known method developed in [13, 14]. Following this method, let us assume $\tau(x) \equiv 0$ for $-\infty < x < 0$ and let us introduce the function

$$u(x) = \int_{-\infty}^{\infty} k(x-s)\,\tau(s)\,ds \tag{3.1}$$

then using the known convolution property, let us give the integro-differential equation (1.13') under the condition (1.14') the form

$$\int_{-\infty}^{\infty} \{c^2 P + i\lambda u(0) - [c^2 + \lambda^2 K(\lambda)]\, T(\lambda)\} e^{-i\lambda x}\, d\lambda = 0 \qquad (0 < x < \infty)$$

where $K(\lambda)$ is defined by (2.3)

$$T(\lambda) = \int_{-\infty}^{\infty} \tau(x)\, e^{i\lambda x}\, dx \tag{3.2}$$

We can write the condition $\tau(x) = 0$ for $-\infty < x < 0$ as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} T(\lambda)\, e^{-i\lambda x}\, d\lambda = 0 \qquad (-\infty < x < 0)$$

Then on the basis of [14], there exist functions $\Phi_+(z)$ and $\Phi_-(z)$ which are holomorphic in the upper and in the lower half-plane, respectively. These functions vanish at infinity and they have the following form on the real axis:

$$\Phi_+(\lambda + i0) = T(\lambda)$$

$$\Phi_-(\lambda - i0) = -[c^2 + \lambda^2 K(\lambda)]\, T(\lambda) + c^2 P + i\lambda u(0)$$

Hence

$$\Phi_+(\lambda + i0) = -\frac{\Phi_-(\lambda - i0)}{c^2 + \lambda^2 K(\lambda)} + \frac{c^2 P + i\lambda u(0)}{c^2 + \lambda^2 K(\lambda)} \tag{3.3}$$

$$(-\infty < \lambda < \infty)$$

where in conformity with (1.14'), we have from (3.2)

$$\Phi_+(0 + i0) = P \tag{3.4}$$

Therefore, the integro-differential equation (1.13') under the condition (1.14') is equivalent to the Riemann boundary value problem (3.3) for a half-plane under the condition (3.4). If the solution of this problem is known, i.e. two functions $\Phi_+(z)$ and $\Phi_-(z)$ possessing the above-mentioned properties have been found, then the solution of (1.13') under the condition (1.14') is determined by

$$\tau(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_+(\lambda + i0)\, e^{-i\lambda x}\, d\lambda \tag{3.5}$$

Since the function $c^2 + \lambda^2 K(\lambda)$ is strictly positive and even on the real axis, the index of the Riemann problem (3.3) equals zero. Therefore, this problem has a unique solution, where the function $\Phi_+(\lambda + i0)$ is expressed in conformity with [14] by the formula

$$\Phi_+(\lambda + i0) = \frac{c^2 P + i\lambda u(0)}{2[c^2 + \lambda^2 K(\lambda)]} +$$

$$\frac{H(\lambda)}{2\pi i[c^2 + \lambda^2 K(\lambda)]} \int_{-\infty}^{\infty} \frac{c^2 P + itu(0)}{H(t)(t-\lambda)} dt \qquad (-\infty < \lambda < \infty)$$

$$H(\lambda) = \sqrt{W(\lambda)} \exp\left[-\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln W(t)\, dt}{t-\lambda}\right]$$

$$W(\lambda) = -[c^2 + \lambda^2 K(\lambda)]$$

The integrals herein should be understood in the Cauchy principal value sense. Substituting the expression for the function $\Phi_+(\lambda + i0)$ into (3.5), we obtain the contact stresses $\tau(x)$ under a semi-infinite gusset, after which the constant $u(0)$ can be determined from (3.1).

It should be noted that the expression $\Phi_+(\lambda + i0)$ obtained is complex in structure. On the other hand, the problem of the effective exact factorization of the function $c^2 + \lambda^2 K(\lambda)$ on the real axis is fraught with great difficulties. Hence, from the practical viewpoint it is convenient to have an approximate factorization of this function. A method of approximate factorization of this function will be proposed here which is based on reducing the Riemann problem (3.3) under the condition (3.4) to an infinite system of linear equations.

To this end, let us represent (3.3) as

$$\Phi_+(\lambda + i0) - \Phi_-(\lambda - i0) = -R(\lambda)\Phi_-(\lambda - i0) + g(\lambda)$$

$$(-\infty < \lambda < \infty)$$

$$R(\lambda) = \frac{1 + c^2 + \lambda^2 K\lambda}{c^2 + \lambda^2 K(\lambda)}, \qquad g(\lambda) = \frac{c^2 P + i\lambda u(0)}{c^2 + \lambda^2 K(\lambda)} \tag{3.6}$$

Then

$$\Phi(z) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{R(s)\Phi_-(s - i0)\, ds}{s - z} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{g(s)\, ds}{s - z}$$

$$\Phi(z) = \begin{cases} \Phi_+(z), & \operatorname{Im} z > 0 \\ \Phi_-(z), & \operatorname{Im} z < 0 \end{cases}$$

where $\Phi_+(z)$ and $\Phi_-(z)$ are the above-mentioned functions. Using the known Sokhotskii-Plemelj formulas, we find

$$\Phi_-(\lambda - i0) = \frac{1}{2} R(\lambda)\Phi_-(\lambda - i0) - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{R(s)\Phi_-(s - i0)\, ds}{s - \lambda} + G(\lambda) \tag{3.7}$$

where

$$G(\lambda) = -\frac{1}{2} g(\lambda) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{g(s)\, ds}{s - \lambda} \tag{3.8}$$

is a known function. Therefore, the Riemann problem (3.3), together with (3.4), is equivalent to the singular integral equations(3.7) in $\Phi_-(\lambda - i0)$, to which should be appended the condition

$$\Phi_-(0 - i0) = 0 \tag{3.9}$$

resulting from (3.4). After the function $\Phi_-(\lambda - i0)$ has been determined from (3.7), the function $\Phi_+(\lambda + i0)$ can be determined from (3.3).

Then, let us turn to new variables by assuming

$$\lambda = \operatorname{tg} t/2, \qquad s = \operatorname{tg} u/2 \qquad (-\pi < t,\ u < \pi) \tag{3.10}$$

Omitting the intermediate calculations, let us write the final form of (3.7)

$$\varphi(t) = {}^1/_2 Q(t)\varphi(t) - \frac{1}{4\pi i}\int_{-\pi}^{\pi} \operatorname{ctg}\frac{u-t}{2}\varphi(u)\,du + h(t) - A \tag{3.11}$$

where

$$\varphi(t) = \Phi_-(\operatorname{tg}{}^1/_2 t - i0), \qquad Q(t) = R(\operatorname{tg}{}^1/_2 t)$$

$$h(t) = G\left(\operatorname{tg}\frac{u}{2}\right), \qquad A = \frac{1}{4\pi i}\int_{-\pi}^{\pi} \operatorname{tg}\frac{u}{2}\,\varphi\left(\operatorname{tg}\frac{u}{2}\right) du$$

Hence, condition (1.16) goes over into the following:

$$\varphi(0) = 0 \tag{3.12}$$

Thus, the singular integral equation (3.7) with Cauchy kernel is reduced, under the condition (3.9), to the singular integral equation (3.11) with the Hilbert kernel under the condition (3.12). Let us note that in conformity with the first formula in (3.6)

$$Q(t) = \frac{1 + c^2 + \operatorname{tg}^2 t/2K(\operatorname{tg} t/2)}{c^2 + \operatorname{tg}^2 t/2K(\operatorname{tg} t/2)} \tag{3.13}$$

Let us seek the solution of the integral equation (3.11) as the Fourier series

$$\varphi(t) = \sum_{m=-\infty}^{\infty} \varphi_m e^{imt} \qquad (-\pi < t < \pi)$$

Simultaneously, let us assume

$$Q(t) = \sum_{m=-\infty}^{\infty} Q_m e^{imt}, \qquad h(t) = \sum_{m=-\infty}^{\infty} h_m e^{imt} \qquad (-\pi < t < \pi)$$

Taking account of the known Hilbert formulas

$$\frac{1}{2\pi i}\int_{-\pi}^{\pi} \operatorname{ctg}\frac{u-t}{2} e^{iku}\,du = \operatorname{sign} k\, e^{ikt} \qquad (k = 0, \pm 1, \pm 2, \ldots)$$

we obtain from (3.11)

$$\varphi_m = h_m \qquad (m = 1, 2, \ldots)$$
$$\varphi_0 = {}^1/_2\psi_0 + h_0 - A$$
$$\varphi_m = \psi_m + h_m \qquad (m = -1, -2, \ldots)$$

$$\psi(t) = Q(t)\varphi(t) = \sum_{m=-\infty}^{\infty} \psi_m e^{imt} \qquad (-\pi < t < \pi)$$

Evidently

$$\psi_k = \sum_{m=-\infty}^{\infty} Q_{k-m}\varphi_m \qquad (k = 0, \pm 1, \pm 2, \ldots)$$

Now, taking account of (3.8), we find $h_m = 0\,(m = 1, 2, \ldots)$. Consequently,

$$\varphi_m = \sum_{m=-\infty}^{\infty} Q_{k-m}\varphi_m + h_m \qquad (m = -1, -2, \ldots) \tag{3.14}$$

$$\varphi_0 = \frac{1}{2} \sum_{m=-\infty}^{\infty} Q_m\varphi_m + h_0 - A$$

Moreover, according to (3.12)

$$\sum_{m=-\infty}^{0} \varphi_m = 0 \tag{3.15}$$

Thus, an infinite system of linear equations (3.14) representing a discrete analog of integral equations on a half-line with kernels dependent on the difference in arguments has been obtained to determine the unknown coefficients $\{\varphi_m\}_{m=-\infty}^{0}$. The theory of such equations has been developed in [13]. It should be noted that this result could have been foreseen a priory if we had taken into account that the transformation (3.10) to the problem of factorizing some function on the real axis reduces to the problem of factorizing the corresponding function in the unit circle.

After some manipulation, (3.14) and (3.15) can be represented as

$$\varphi_k^* = -\frac{1}{Q_0} \sum_{m=1}^{\infty}{}' Q_{k-m}\varphi_m^* + h_k^* \qquad (k = 1, 2, \ldots) \tag{3.16}$$

$$\varphi_0^* = \frac{1}{2} \sum_{m=0}^{\infty} Q_m\varphi_m^* + h_0^* - A, \qquad \sum_{m=0}^{\infty} \varphi_m^* = 0 \tag{3.17}$$

where

$$\varphi_k^* = \varphi_{-k}, \quad h_k^* = -(h_{-k} + Q_k\varphi_0^*)\,Q_0^{-1} \qquad (k = 1, 2, \ldots)$$
$$\varphi_0^* = \varphi_0, \qquad h_0^* = h_0$$

The prime to the right of the summation sign in (3.16) means that the term with subscript $m = k$ is omitted in this sum.

Let us investigate the question of regularity of the infinite system (3.16). To this end, let us form the sums

$$S_k = |Q_0|^{-1} \sum_{m=1}^{\infty}{}' |Q_{k-m}| = |Q_0|^{-1} \sum_{p=1-k}^{\infty}{}' |Q_p| \quad (k = 1, 2, \ldots)$$

where the term with zero subscript is omitted in the last sum. It is easy to see that

$$S_k \leqslant |Q_0|^{-1} \sum_{p=-\infty}^{\infty}{}' |Q_p| \qquad (k = 1, 2, \ldots)$$

Let us note that in conformity with (3.13)

$$Q_p = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(t)\cos pt\, dt \quad (p = 0, \pm 1, \pm 2, \ldots)$$
$$P(t) = [c^2 + \mathrm{tg}^2\, {}^1\!/\!_2 t K(\mathrm{tg}\, {}^1\!/\!_2 t)]^{-1}$$

According to the known Bochner-Khinchine theorem, the function $[c^2 + \lambda^2 K(\lambda)]^{-1}$ is a positive-definite function on the real axis. Therefore, the function $P(t)$ is a positive-definite function in the segment $[-\pi, \pi]$, whereupon its Fourier coefficients are nonnegative. The above permits the assertion that

$$S_k \leqslant (P(0) - Q_0) Q_0^{-1} \qquad (k = 1, 2, \ldots)$$

Let us require that $P(0)/Q_0 < 2$ or

$$c^2 Q_0 > 1/2 \tag{3.18}$$

Upon compliance with condition (3.18), the infinite system (3.16) is completely regular, and therefore, its solution can be found by the method of reduction or successive approximations. After the coefficients $\{\varphi_m^*\}_{m=1}^{\infty}$ have been found, the coefficients φ_0^* and A can be determined from (3.17).

Let us show that for sufficiently large c^2 the condition (3.18) is satisfied. Indeed

$$c^2 Q_0 = \frac{c^2}{\pi} \int_0^{\pi} \frac{dt}{c^2 + \operatorname{tg}^2 t / 2K(\operatorname{tg} t/2)} > \frac{c^2}{\pi} \int_0^{\pi-\varepsilon} \frac{dt}{c^2 + \operatorname{tg}^2 K(\operatorname{tg} t/2)}$$

where ε is a small positive number within the interval $(0, \pi/2)$. Furthermore, let us introduce the notation

$$M = \max_{0 \leqslant t \leqslant \pi - \varepsilon} \left[c^2 + \operatorname{tg}^2 \frac{t}{2} K\left(\operatorname{tg} \frac{t}{2}\right)\right]$$

Then

$$c^2 Q_0 > \frac{c^2}{\pi} \int_0^{\pi-\varepsilon} \frac{dt}{c^2 + M} = \frac{c^2}{c^2 + M}\left(1 - \frac{\varepsilon}{\pi}\right)$$

It hence follows that for sufficiently large c^2 and sufficiently small ε the mentioned condition is satisfied.

Let us note that when $c^2 = 0$, the integro-differential equation (1.13') or equation (1.15) describes the appropriate contact problem for an absolutely rigid semi-infinite gusset, i.e. for a semi-infinite stamp. In this case, the condition (3.18) will not hold, and therefore, this problem requires a separate investigation.

4. Case of a finite gusset. Now, let us turn to the solution of the integro-differential equation (1.17') under the boundary conditions (1.18'). By using (1.19), after some manipulations we arrive at the singular integro-differential equation

$$\int_{-1}^{1} \left[\frac{1}{s - x} + V(x - s)\right] \varphi'(s)\, ds = ac^2 \varphi(x) \tag{4.1}$$

under the boundary conditions

$$\varphi(-1) = 0, \qquad \varphi(1) = P \tag{4.2}$$

where in conformity with (1.20)

$$V(x) = \frac{dr(ax)}{dx} = \frac{d}{dx}\left[\frac{\delta}{\sqrt{\delta^2 + a^2x^2}} K\left(\frac{\delta}{\sqrt{\delta^2 + a^2x^2}}\right)\right] + \frac{1}{x} \qquad (-1 < x < 1)$$

Here

$$\varphi(x) = \int_{-1}^{x} \tau^*(s)\, ds, \quad \tau^*(x) = a\tau(ax) \qquad (-1 < x < 1)$$

where the contact stress under the gusset is now determined by the formula

$$\tau(x) = (1/a)\varphi'(x/a) \qquad (-a < x < a)$$

Following [6], let us assume

$$\varphi'(x) = \frac{x_0 + \sum_{n=1}^{\infty} x_n T_n(x)}{\sqrt{1-x^2}}$$

where $T_n(x) = \cos(n \arccos x)$ are Chebyshev polynomials of the first kind. Then, by proceeding completely analogously to the exposition in [6], we show that the solution of the integro-differential equation (4.1) under the boundary conditions (4.2) reduces to the solution of the following infinite system in the coefficients $\{x_n\}_{n=1}^{\infty}$:

$$x_m = \frac{2P}{\pi^3} a_m - \frac{2}{\pi^2} \sum_{n=1}^{\infty} [K_{m,n}^{(1)} + K_{m,n}^{(2)}] x_n \qquad (m = 1, 2, \ldots) \tag{4.3}$$

$$a_m = \int_0^{\pi} [ac^2(\pi - t) - \vartheta_0(t)] \sin mt \sin t \, dt \qquad (m = 1, 2, \ldots)$$

$$K_{m,n}^{(1)} = \frac{c^2}{n} \int_0^{\pi} \sin nt \sin mt \sin t \, dt \qquad (m, n = 1, 2, \ldots)$$

$$K_{m,n}^{(2)} = \int_0^{\pi} \vartheta_n(t) \sin mt \sin t \, dt$$

$$\vartheta_n(t) = \int_0^{\pi} V(\cos t - \cos u) \cos nu \, du \qquad (n = 0, 1, 2, \ldots)$$

An investigation of the infinite system (4.3) in the case of the one kernel $K_{m,n}^{(1)}$ is contained in [6]. Let us show that the addition of a new kernel $K_{m,n}^{(2)}$ to the former does not violate the regularity of the initial infinite system in the sense of its quasi-complete regularity.

Indeed, on the basis of Sect. 1 relative to the properties of the function $r(x)$, we can write

$$K_{m,n}^{(2)} = -\frac{1}{n} d_{m,n} \qquad (m, n = 1, 2, \ldots)$$

$$d_{m,n} = \int_0^{\pi}\int_0^{\pi} d(u, t) \sin t \sin mt \sin nu \, du \, dt$$

$$d(u, t) = \frac{d}{du}[V(\cos t - \cos u)]$$

Therefore

$$S_m = \sum_{n=1}^{\infty} |K_{m,n}^{(2)}| - \sum_{n=1}^{\infty} \frac{1}{n} |d_{m,n}|$$

Let us note that the coefficients $\{d_{m,n}\}_{m,n=1}^{\infty}$ are Fourier coefficients of the continuous function of two variables $d(u, t) \sin t$ in the complete system of functions $\{\sin nu \sin mt\}_{m,n=1}^{\infty}$ orthogonal in the square $0 < t, u < \pi$. Then by virtue of the Bessel inequality, the following two series converge

$$\sum_{m,n=1}^{\infty} |d_{m,n}|^2$$

Therefore, the series [15]

$$\sum_{m=1}^{\infty} D_m, \qquad D_m = \sum_{n=1}^{\infty} |d_{m,n}|^2$$

also converges, i.e. at least

$$D_m = O(m^{-(1+\varepsilon)}) \quad m \to \infty \tag{4.4}$$

where ε is a small positive number. Thus

$$S_m = \sum_{n=1}^{\infty} \frac{1}{n} |d_{m,n}| \qquad (m = 1, 2, \ldots)$$

Hence, by using the Cauchy-Buniakowski inequality we obtain

$$S_m \leqslant \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^2}} \sqrt{\sum_{n=1}^{\infty} |d_{m,n}|^2} = \frac{\pi}{\sqrt{6}} \sqrt{D_m}$$

Taking (4.4) into account we have

$$S_m = O(m^{-(1+\varepsilon)/2}) \quad m \to \infty$$

which indeed proves the assertion made above.

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VARIATIONAL METHODS OF CONSTRUCTING MODELS OF SHELLS (*)

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The purpose herein is to derive the relationships of the theory of elastic shells from the variational equation of the mechanics of continuous media in the general case of physically and geometrically nonlinear models. The examination of this question is interesting in connection with the fact that all the hypotheses acquire the most compact and explicit formulation in the variational approach, and a logical basis appears for the comparison and estimation of the various models proposed in the theory of shells. Moreover, the shell models yield an interesting illustration of models of continuous media in which there are firstly higher derivatives, and secondly, internal degrees of freedom originate, as will be seen later. The appearance of the internal degrees of freedom requires the establishment of additional equations, in addition to the ordinary equations of mechanics, in order to determine new parameters, and to raise the order of the differential equations – additional boundary conditions and conditions on discontinuities. These relationships have been obtained by using methods developed for arbitrary models of continuous media with internal degrees of freedom and with higher derivatives in [1, 2]. Let us note that the extension of the theory to inelastic shells is associated only with complicating the functional δW^* in (1.1) and adding new degrees of freedom due to plastic deformations, viscous deformations, etc. Only the general part of the theory is contained herein. Specific shell models will be examined separately.

1. Variational equation in the theory of elastic bodies. The fundamental relationships of the theory of elastic bodies can be obtained from the variational equation [1 – 3]

$$\delta \iint_{tV} \Lambda \, d\tau \, dt + \delta W^* + \delta W = 0 \tag{1.1}$$

where the Lagrangean Λ and the functional δW^* are the given quantities, and δW is an integral of a linear combination of the variations in the displacements over the bound-

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